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LETTER TO THE EDITOR

Regular versus irregular Laplacian growth: multifractal spectroscopy

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Abstract. We analyse the effect of tip stability on the multifractality of interfacial patterns by applying the method of noise reduction to a random Laplacian growth on the square lattice. While the distribution of perimeter sites over the harmonic measure is smooth in the case of random fractal objects with tip splitting, separate peaks appear if the tips are stable. These peaks can be identified with the subset of growth sites corresponding to the tips. We demonstrate the consequences of such behaviour in the multifractal spectrum.

Interfacial patterns developed by Laplacian processes (Langer 1980, Bensimon *et al* 1986) are known to exhibit different behaviour depending on the amount of anisotropy present during the growth (Kessler *et al* 1986). The tips of the fingers or elongated parts which emerge due to the instability of the interface are split if anisotropy is too small. In the presence of sufficient randomness the sequence of instabilities caused by tip splitting and screening leads to a random fractal object. On the other hand, a large enough anisotropy stabilises the tips and the resulting pattern is regular (Kertèsz 1987, Vicsek 1987).

Random Laplacian models like diffusion-limited aggregation (DLA) (Witten and Sander 1981, 1983) or dielectric breakdown model (DBM) (Niemeyer *et al* 1984) have been widely studied examples of fractal growth. If an underlying lattice structure is present, the pattern is a result of the interplay between the anisotropy and fluctuations. The fluctuations tend to suppress anisotropy and therefore, at least for small sizes, the tip splitting character of the growth is dominant. However, as the size of the system increases, the weight of the fluctuations decreases and the tip stabilising effect of the anisotropy can develop. In fact, DLA clusters on the square lattice have been shown to cross over from the random fractals to patterns having the more regular overall shape of dendrites with stable tips in the directions of the lattice axes (Meakin 1986). During this crossover the value of the fractal dimensionality (or exponent of radius of gyration) changes from ~ 1.7 to ~ 1.5.

In order to control the amount of fluctuations in random Laplacian growth the method of noise reduction was introduced independently by Tang (1985) and Szép *et al* (1985). The algorithm for DLA is as follows: instead of adding a particle to the

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aggregate if the random walker coming from 'infinity' hits a perimeter, one occupies a growth site only if it is hit by m independent random walkers where m is the noise reduction parameter. As was first pointed out by Kertèsz and Vicsek (1986), the effect of noise reduction on square lattice DLA clusters is that the asymptotic shape of the pattern is reached much earlier than in pure DLA: while millions of particles are needed to see the dendritic structure if $m=1$, it is well developed already for very small sizes (a hundred particles) even at small values of m ($m=4$). Thus noise reduction is a simple tool to tune the anisotropy in growth of clusters of a given size.

The difference between a random fractal cluster and a regular dendritic pattern is visually apparent and we are now interested in the consequences of this difference on the quantities characterising such objects. The multifractal analysis of random fractals has led to important new insight into their geometry and DLA (or DBM) clusters are known to obey multifractal behaviour with respect to the growth probability distribution as a measure (see Coniglio (1986) and Stanley and Meakin (1987) for reviews). Amitrano *et al* (1986) showed that by changing the parameter η in the DBM model not only the fractal dimensionality changes but there is also a drastic change in the distribution of the number of growth sites. Here we undertake a somewhat similar project related to the differences due to the change of the parameter m: the aim of the present letter is to investigate the effect of tip stability on the quantities characterising multifractality.

Let us first summarise briefly the formalism describing the multifractal behaviour of the clusters (Halsey *et al* 1986a, b). Associate a measure p_i at site i. Usually p_i is the growth probability, the probability that site i becomes part of the cluster at the next step. We consider the following moments:

$$Z(q)=\sum p_i^q \propto L^{-\tau(q)} \tag{1}$$

where L is the linear extent of the cluster and $\tau(q)$ are the critical exponents. Equation (1) can be rewritten by defining the density distribution $n(\ln(p))$ with p being the continuous parameter describing the growth probabilities:

$$Z(q)=\int n(x)\,\mathrm{e}^{qx}\,\mathrm{d}x. \tag{2}$$

The integral in (2) can usually be evaluated by the steepest-descent method, i.e. the integral can be substituted by the maximum of the integrand. In this way, for each q we have $x^*=\ln p^*$ defined by

$$\frac{\mathrm{d}}{\mathrm{d}x}\ln n(x)=q. \tag{3}$$

We can now write the following scaling *ansatz*:

$$p^*\propto L^{-\alpha(q)} \tag{4}$$

$$n(x^*)\propto L^{f(q)} \tag{5}$$

and from (2)

$$\tau(q)=q\alpha(q)-f(q) \tag{6}$$

$$\frac{\mathrm{d}}{\mathrm{d}q}\tau(q)=\alpha(q). \tag{7}$$

Thus the fundamental quantity to calculate is $n(x)$ as a function of L from which $\tau(q)$ and finally $f(\alpha)$ can be obtained.

In a numerical determination of $n(x)$ one has to simulate many samples of the fractal object and take their averages. In order to see the effect of tip stabilisation on $n(x)$ and $f(\alpha)$ we implemented the following algorithm. We consider the DBM and calculate the probabilities p_i by using the Green function method on the square lattice (Morita 1971). Having found the probabilities we make several trials and count how many times a given growth site is chosen. If this number reaches the noise reduction parameter m we occupy the site. For the new configuration we again calculate the probabilities but let the counters remain unchanged. Note that in this case p_i is the probability for a given growth site to be chosen and is associated to the harmonic measure; it is not the growth probability, i.e. the probability that a site becomes part of the aggregate at the next step: only for $m = 1$ do the two coincide. In the following we will consider the moments of p_i, namely we will associate the harmonic measure to each site. Therefore the multifractal analysis will be made in terms of the harmonic measure. The application of noise reduction to DBM (Nittmann and Stanley 1986) is known to lead to results different from the DLA due to the differences in the boundary conditions (Ball 1986). However, our Green functional approach leads to configurations qualitatively similar to the DLA case.

The Green function method has the advantages that no outer boundary condition is needed at finite distance from the aggregate and the small p_i can also be calculated with arbitrary accuracy. The disadvantage of this method is that one cannot go to large sizes: the maximum number of particles we could deal with was of the order of 100. However, since noise reduction is already effective at such sizes, one can expect relevant results.

Our main findings are summarised in figure 1. $n(x)$ is usually a quasi-continuous smooth function because it is an averaged quantity (Amitrano *et al* 1986). However, in the case of stable tips a different behaviour is observed. At the tips, which are the 'hottest' parts of the aggregates, there are no more fluctuations and therefore $n(x)$ has a sharp separate peak corresponding to the highest growth probabilities. In fact, we can see that the peak occurs parallel with the stabilisation of the tip and the sharpness of the peak characterises its stability. In other words, the peak at the highest probabilities can be identified with a well defined subset of the cluster: a situation similar to spectroscopy where sharp peaks correspond to eigenstates (stable tips) and the width to the inverse lifetime (stability). According to this picture further peaks should separate off from the 'continuum' if the shape of the tip is more fixed and this is observed if m is increasing for a given number of particles (figure 1(c), (d)).

Let us now consider how the multifractal formalism (1)–(7) has to be changed in the presence of a sharp peak in $n(x)$ (the generalisation to more than one peak is straightforward). For this purpose we suppose that $n(x)$ consists of two parts:

$$n(x) = n(x_t)\delta(x - x_t) + n_s(x) \tag{8}$$

where $x_t = \ln p_t$ denotes the maximum value of x where the peak is located and n_s is the smooth part of n. If we now evaluate the integral (2) we get

$$Z(q) = L^{f_t - q\alpha_t} + L^{f_s(q) - q\alpha_s(q)} \tag{9}$$

where (2), (4) and (5) were used and we assumed that relations similar to (4) and (5) are valid for the set characterised by the peak. Since one is interested in the large L limit, the contribution to $Z(q)$ in (9) comes from the term which has a larger exponent.

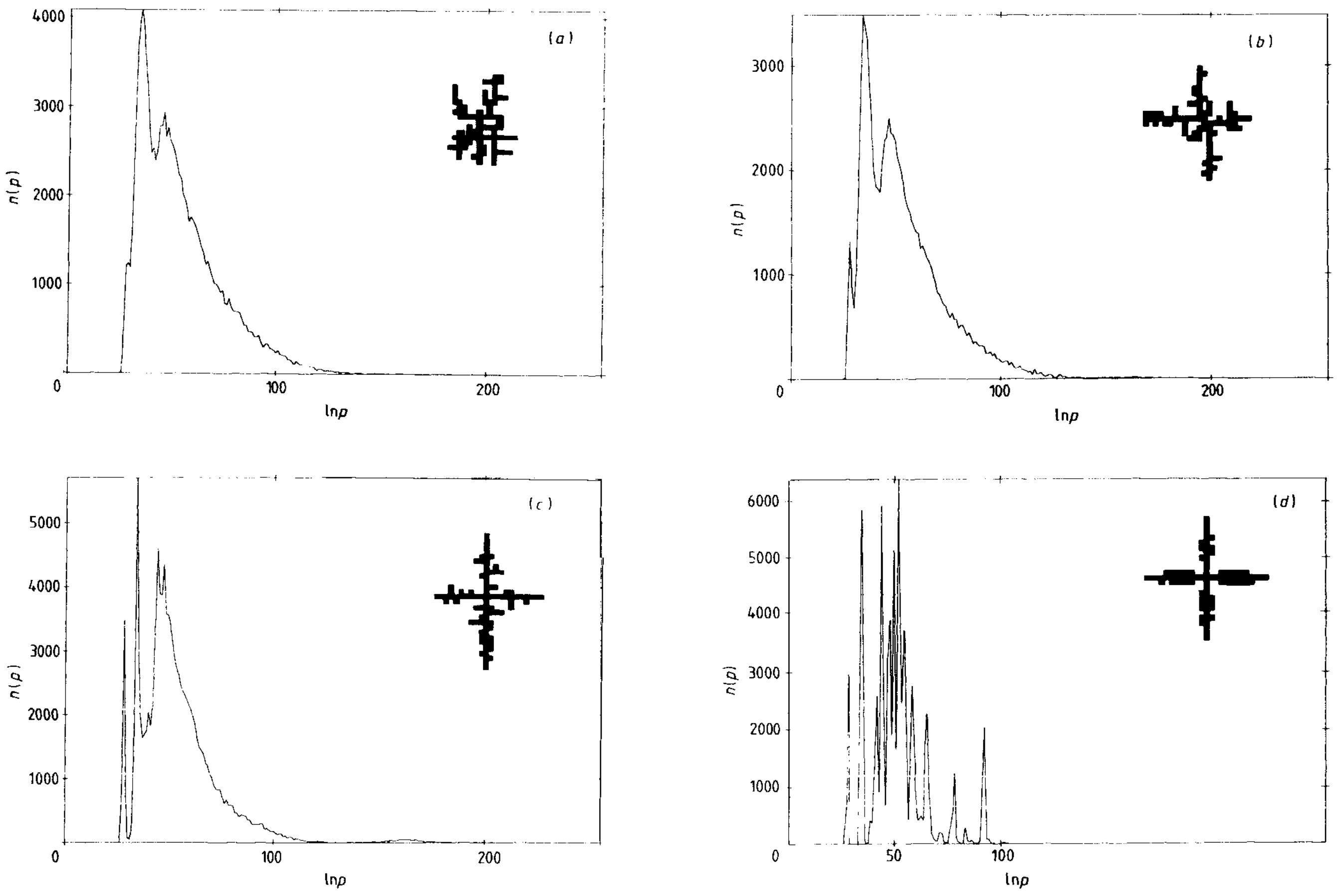
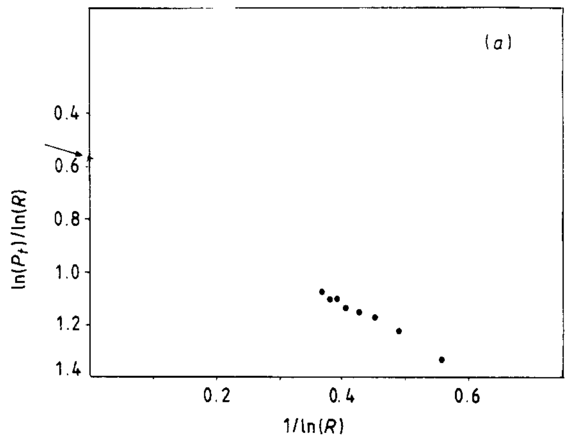


Figure 1. Distribution of the number of growth sites n as a function of $-x = -\ln(p)$ for different values of noise reduction. The number of particles in the clusters always 90. (a) $m = 2$, 1500 configurations; (b) $m = 4$, 1200 configurations; (c) $m = 20$, 1300 configurations; (d) $m = 400$, 920 configurations. Characteristic clusters shown as inserts. Peaks in $n(x)$ can be identified with definite stable subsets of the clusters.

One possibility is that $\tau_t(q)$ takes over at a critical value of q which is the solution of the equation $\tau_t(q_c) = \tau_s(q_c)$. The result is a kink in the full exponent $\tau(q)$, i.e. a separate point (α_t, f_t) occurs on the $f(\alpha)$ plot. However, another possibility is, of course, that there is no discontinuity in $f(\alpha)$ and the point (α_t, f_t) fits well onto the $f(\alpha)$ curve.

The identification of the peak in $n(x)$ with a subset enables us to calculate the corresponding exponents immediately from the L dependence of the position x_t of the peak (for α_t) and of the area (for f_t). This extrapolation is shown in figure 2 for $m = 20$. Our observations can be summarised as follows. α_t and f_t agree within the error bars with the minimum value of α and f obtained from the $f(\alpha)$ curve (figure 3). We observe that, whenever the first peak is well defined, the corresponding fractal dimensionality f_t is very close to zero; in addition, the number of growth sites belonging to the singularity characterised by this peak is about four. This can be understood, since the contributions to the first peak come from the subset of growth sites with highest measure, i.e. from the four stable tips of the main stems. The analysis of the second peak leads to similar results: the f and α exponents are close to the values obtained for the first peak. Taking into account that the variation of f is rapid in the



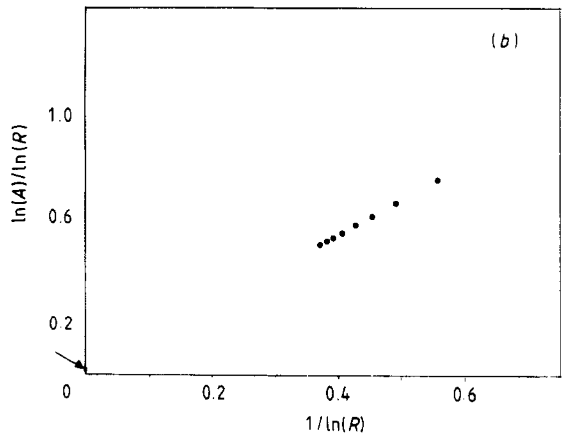


Figure 2. Extrapolation of f_t and α_t from the position and area of the first peak in $n(x)$ for $m = 20$. The plots show (a) $\ln p_t / \ln R$ and (b) ln(area)/ln R as a function of $1/\ln R$. The intercepts, calculated through the best fit of the data, are marked with arrows on the graphs.

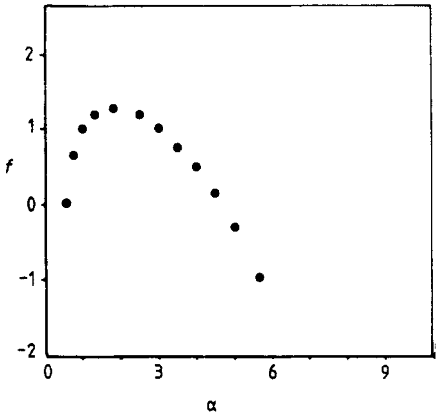


Figure 3. $f(\alpha)$ for $m = 20$. The contribution from the separate peak in $n(x)$ (see figure 1(c)) practically coincides with the first point of the curve (check with the values for α and f taken from figure 2(a) and (b)).

neighbourhood of the minimum value of α we think that in our case the whole $f(\alpha)$ curve turns out to be continuous. A sharp kink in $\tau(q)$ can probably be excluded.

In summary we have demonstrated that the multifractal analysis of the problem of tip stability in Laplacian growth is very useful; in particular the peaked structure of $n(x)$ enables us to identify the subsets of the clusters which are characteristic for the stable tips. Since the stability means less fluctuations, the sites on the stable tips contribute to well defined probabilities and are thus similar to eigenstates in spectroscopic problems. We think that behaviour similar to the one described above may occur in other related problems and especially in non-linear dynamics.

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